EXACT SOLUTIONS OF AXISYMMETRIC FLOWS OF AN IDEAL FLUID

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The equation for the Stokes' stream function ψ in steady potential axisymmetric flows of incompressible fluids in cylindrical coordinates x, y, θ takes the form:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{y}\frac{\partial\psi}{\partial y}=0 \qquad (y>0) \tag{1}$$

If we set $\psi=\sqrt{y}\,\psi^0$, we obtain

$$\frac{\partial^2\psi^\circ}{\partial x^2}+\frac{\partial^2\psi^\circ}{\partial y^2}-\frac{3\psi^\circ}{4y^2}=0 \tag{2}$$

We will seek solutions in the form [1]:

$$\psi^\circ=\Phi_0(x,\ y)+\sum_{k=1}^{\infty}\Phi_k(x,\ y)\,f_k(y) \tag{3}$$

where Φ_0, Φ_k are arbitrary harmonic functions. Equation (2) then becomes:

$$-\frac{3}{4y^2}\Phi_0+\sum_{k=1}^{\infty}\left[2\frac{\partial\Phi_k}{\partial y}f_k'+\Phi_k\left(f_k''-\frac{3}{4y^2}f_k\right)\right]=0 \tag{4}$$

On the functions f_k and Φ_k we impose the following conditions

$$f_k''-\frac{3f_k}{4y^2}=f_{k+1}', \qquad 2\frac{\partial\Phi_k}{\partial y}=-\Phi_{k-1} \tag{5}$$

so that (4) takes the form:

$$-\left(\frac{3}{4y^2}+f_1'\right)\Phi_0+\Phi_k f_{k+1}'=0, \qquad \lim_{k\to\infty}\Phi_k f_{k+1}'=0 \tag{6}$$

Consequently,

$$f_1=-\frac{3}{4}\int_{\infty}^{y}\frac{dy}{y^2}=\frac{3}{4y}, \qquad f_{k+1}=\frac{df_k}{dy}-\frac{3}{4}\int_{\infty}^{y}\frac{f_k}{y^2}dy, \qquad \Phi_k=-\frac{1}{2}\int_{\infty}^{y}\Phi_{k-1}\,dy \tag{7}$$

The formula for f_k follows easily

$$f_k = (-1)^k \frac{k!\,C_k}{y^k}, \qquad C_k = \frac{C_{k-1}(k + 1/2)(k - 3/2)}{k^2}, \qquad C_0 = 1 \tag{8}$$

Let us introduce the complex potential $W(z) = \phi_0(x, y) + i\,\Phi_0(x, y)$, $(z = x + iy)$. Then

$$\Phi_0 = \operatorname{Im} W(z), \qquad \Phi_k = \frac{(-1)^k}{(k-1)!\,2^k} \operatorname{Im} \int_0^z (z-\zeta)^{k-1} W(\zeta)\, d\zeta \tag{9}$$

$$\psi^0 = \operatorname{Im} \left\{ W(z) - \int_0^z W(\zeta) \sum_{k=1}^{\infty} \frac{(-1)^k (z-\zeta)^{k-1}}{2^k (k-1)!} f_k(y)\, d\zeta \right\} \tag{10}$$

Recognizing the character of hypergeometric series in the variations of C_k and (8), we obtain

$$\psi^0 = \operatorname{Im} \left\{ W(z) - \int_0^z W(\zeta) \frac{d}{d\zeta} H\left(\frac{3}{2}, -\frac{1}{2}, 1, \frac{z-\zeta}{2y}\right) d\zeta \right\} \tag{11}$$

Setting $W(0) = 0$, we arrive at the desired solution of (1):

$$\psi = \sqrt{y}\, \operatorname{Im} \left\{ \int_0^z \frac{dW(\zeta)}{d\zeta} H\left(\frac{3}{2}, -\frac{1}{2}, 1, \frac{z-\zeta}{2y}\right) d\zeta \right\} \tag{12}$$

where $W(\zeta)$ is an arbitrary function. The solution of the axisymmetric problem of incompressible fluids has been related to the complex potential solution of the two-dimensional problem. It is possible to obtain more general solutions when we consider indefinite integrals in (7).

BIBLIOGRAPHY

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